A New application of Estimating Functions to Point, Variance and Interval Estimation for Simple and Complex Surveys

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1. Introduction

The method of estimating function (EF), also popularly known as quasi-likelihood (QL) method of estimation, is commonly used in practice for inference with models under a semi-parametric framework when only the first two moments are specified for the model error. This semi-parametric framework is particularly relevant for survey data because it is typically not feasible to compute beyond first and second order inclusion probabilities under the sampling design. (For complex designs, the variance of the EF is generally simplified by expressing it as a variance under simple random sampling with a multiplicative adjustment by the design effect assumed to be approximately known.) The standardized EF provides a natural pivot with asymptotic normal distribution for testing and interval estimation purposes. It is known that the above normal approximation for EF holds well even for moderate sample sizes unlike that for the estimator obtained after Taylor linearization of the EF; see e.g., McCullagh (1991). In this paper, we exploit the above stability property of the normal approximation to the distribution of EF to propose a unified solution with the goal of possibly improving variance and interval estimation for various problems whenever the QL method doesn't perform well due to small sample sizes or is not easily applicable due to nonsmooth EF. The QL method provides point estimate (PE) via solution of the EF set to zero, variance estimate (VE) via linearization of the EF if nonlinear, and interval estimate (IE) via approximate normality of the QL estimator.

In fact, the question of improving the QL interval estimation for the problem mentioned above has been addressed in the literature by specially designed methods in order to reduce coverage bias (typically undercoverage), but that of improving QL variance estimation has not, however, been addressed. It turns out that the VE under QL generally behaves well but sometimes may be subject to instability for small samples (in the sense that the coefficient of variation of VE may be high), or may not be available via Taylor linearization when dealing with non-smooth EF. We show that while the proposed method with suitable choices of EF is expected to give results generally at least at par with the existing improved interval estimation methods, it is also expected to provide reasonable VE when the QL method may have problems. Examples of existing IE as an alternative to QL include Wilson's interval for estimating proportions, Fieller's interval for estimating ratio of means, Woodruff's interval for estimating quantiles, and again Fieller's interval for estimating mean response and dose levels in bioassay. The proposed method, termed as the method of randomly recentered estimating equations (RREE), can produce a class of VE and IE for existing problems by using a corresponding class of alternate EFs. A special case of RREE was introduced earlier by Singh and Dochitoiu (2005) in the context of estimating proportions. The RREE method basically consists of creating replicate parameter estimates by solving the estimating equations centered at random values drawn from the pivotal normal distribution of the standardized EF. The usual PE of model parameters under QL corresponds to the center of zero. The Monte Carlo distribution of parameter estimates so obtained, termed as the EF-confidence distribution of parameters, is used to compute new variance and interval estimates. As a byproduct, an alternative point estimate is also obtained by the mean of the EF-confidence distribution. Although this PE is not really needed as it behaves similarly to that under QL, it is interesting nevertheless, and completes the set of compatible solutions to the three estimation problems of PE, VE, and IE in the sense that all the three are obtained as mean, variance, and lower and upper boundaries of IE from a common distribution, namely, the EF-confidence distribution. This is analogous to the QL method where all the three come from the approximate normal distribution of the estimator. Such a compatibility property is appealing in practice although not necessary under the flexibility of the frequentist approach unlike the use of the posterior distribution of the parameter of interest under the Bayseian approach which has built-in compatibility.

It is interesting to note that the RREE method is somewhat related to the EF-bootstrap method of Hu and Kalbfleish (2000) in that the recenter draws from the normal pivotal distribution mimic the draws from the bootstrap distribution of the EF. The EF-bootstrap method, proposed mainly for interval estimation, tends to make more use of the sample data than the RREE, and is expected to perform quite well in comparison to RREE for smaller sample sizes with respect to coverage properties of IE. However, RREE uses a more stable pivotal distribution and therefore, the corresponding VE might be expected to perform favorably for smaller sample sizes. Moreover, unlike EF-Bootstrap, which requires independence of elementary EFs (a condition often violated in survey sampling) whose linear combination defines the EF, the RREE method only requires approximate normality of the EF regardless of the independence of elementary EFs.

The organization of this paper is as follows. In Section 2, the general problem of estimation, corresponding EF (or QL) method of estimation followed by a motivation of the proposed method are presented. Section 3 contains the proposed method of RREE along with a theoretical justification. Next, in Section 4, the alternative method of EF-bootstrap and its comparison with RREE are considered. This is followed in Section 5 by examples of RREE for different problems such as that of estimating proportions, ratio of means, quantiles, and logit model parameters. Section 6 presents a small simulation study for estimating ratio of means, and compares the two alternate methods RREE and EF-Bootstrap in terms of how they might overcome some of the deficiencies of the commonly used QL method for small samples with respect to bias and variance of the point and variance estimators, and average length and coverage probability of the interval estimator. Finally, Section 7 contains summary and concluding remarks.

2. **Quasi-likelihood Method of Estimation and Motivation of the Proposed Method**

Let θ denote a finite population or superpopulation parameter of interest which is assumed to be scalar for simplicity. In the following, we will use matrix notation as much as possible even for scalar parameters in the interest of making an easy transition from scalar to the vector parameter case. Suppose we have a random sample of n observations $\boldsymbol{y}$ and we want a frequentist solution to the estimation problem of θ with triple goals, namely, point estimate (PE), variance estimate (VE) or mean square estimate (ME), and interval estimate (IE) denoted respectively as $\hat{\theta}$, $v(\hat{\theta})$, and $[L_{\alpha/2}(\hat{\theta}), U_{\alpha/2}(\hat{\theta})]$ with equal tailed lower and upper end points for a total confidence of $1-\alpha$. Note that it is the ME that is more relevant as a measure of uncertainty than the VE but in practice we use VE as a substitute for ME because the bias is taken as zero either because the estimate is approximately unbiased or that the bias estimate is not available but deemed as negligible. In Section 6 on the empirical study, we consider performance of $v(\hat{\theta})$ with respect to the true MSE and not the true variance. Our goal in this paper is to have estimates (PE, VE, and IE) with the properties of consistency and unbiasedness for large n but good performance for moderate or small n.

Consider an estimator $\hat{\theta}$ which is obtained as a solution of the estimating equation $\psi(y,\theta)=0$ where the EF $\psi(y,\theta)$ with mean 0 is a linear combination of elementary EFs g_i 's with mean 0. For example, for the common mean parameter θ, g_i could be taken as $y_i-\theta$ and $\psi(y,\theta)$ as $\sum_{i=1}^{n} g_i$. Note that the EF need not be a continuous function of θ, and the estimator $\hat{\theta}$ may not have a closed form. We will assume that if $\psi(y,\theta)$ is not a continuous function of θ, it can be approximated by a continuous function to first order, i.e., error in approximating $n^{-1}\psi(y,\theta)$ is of order $o(1/\sqrt{n})$ for large n. For example, the EF for quantiles involves empirical distribution function which is a discontinuous function of θ but can be approximated by a continuous function made up of piecewise linear segments. In the following, it is also assumed that the EF is a one-to-one function of θ, and that for large n, we have the first order representation,

$$\psi(y,\theta)=J_{\psi}(\hat{\theta}-\theta)+o_p(\sqrt{n}) \tag{2.1}$$

where in matrix notation J_{ψ} is analogous to the observed information matrix $(\partial\psi/\partial\theta')$ if the EF is differentiable with respect to θ or is defined by a suitable linearization as in the case of Bahadur representation for EFs for quantiles. We also assume that under a suitable Central Limit theorem (see e.g., Binder, 1983 for regularity conditions when estimating a finite population parameter), the EF is approximately normal; i.e.,

$$\psi(y,\theta)\sim_{approx} N(0,V_{\psi}(\theta)) \tag{2.2}$$

where $V_{\psi}(\theta)$ is the covariance of the EF $\psi(y,\theta)$ that may depend on θ. The covariance $V_{\psi}(\theta)$ under complex designs can be obtained after adjustment via the design effect of the expression under the working assumption of simple random sampling or that of the ignorable design for the superpopulation model under consideration. It follows that the estimator $\hat{\theta}$ is approximately normal; i.e.,

$$\hat{\theta}-\theta\sim_{approx} N(0,\Sigma_{\theta}) \tag{2.3}$$

where $\Sigma_\theta = J_\psi^{-1}(\theta)V_\psi(\theta)(J_\psi^{-1}(\theta))'$ assuming that the matrix $J_\psi(\theta)$ is nonsingular. The PE $\hat{\theta}$, VE $\Sigma_\theta\big|_{\theta=\hat{\theta}}$, and the normality based IE with $\Sigma_\theta\big|_{\theta=\hat{\theta}}$ comprise the usual frequentist solutions to the estimation problem and fall under the broad category of QL method of estimation.

We can now motivate the proposed method as follows. It is known (see e.g., McCullagh, 1991) that for interval estimation, it is better to use the method of test inversion based on the EF-based pivotal than that based on the estimator. In other words, it is better to use $H_\psi^{-1}(\theta)\psi(y,\theta)$ than $(J_\psi^{-1}(\hat{\theta})H_\psi(\hat{\theta}))^{-1}(\hat{\theta}-\theta)$ where H_ψ is the Cholesky root of V_ψ, i.e., $V_\psi = H_\psi H_\psi'$, and $J_\psi^{-1}H_\psi$ is the Cholesky root of Σ_θ, i.e., $\Sigma_\theta = (J_\psi^{-1}H_\psi)(J_\psi^{-1}H_\psi)'$. The main reason for the distribution of the EF-based pivotal being closer to normality is that the EF is typically a sum of independent or weakly dependent elementary EFs with mean zero whereas the estimator $\hat{\theta}$ is typically a nonlinear function of θ or may not even have a closed form. It might be of interest to note that the basic idea underlying RREE occurred in the context of a hierarchical Bayes application to the small area estimation problem. It had to do with the anticipated superior performance of using EF-based pivotal compounded with the prior to select candidates from the proposal distribution to that of compounding the distribution of the maximum likelihood estimator with the prior. Thus, for a flat prior, one can draw i.i.d. random recenters ε_r, $r = 1,...,R$, from the standard normal, and then solve $H_\psi^{-1}(\theta)\psi(y,\theta) = \varepsilon_r$ to obtain $\tilde{\theta}_r$ as candidates. The empirical or the Monte Carlo distribution $\{\tilde{\theta}_r\}_{1\le r\le R}$ can then be used to derive properties of the proposal posterior distribution of θ. This led to the RREE method defined below under a semi-parametric frequentist framework which is applicable when the likelihood is not specified.

3. Randomly Recentered Estimating Equations (RREE): the proposed method

Consider the empirical distribution $\{\tilde{\theta}_r\}_{1\le r\le R}$ defined in the previous section as an approximate distribution of $\hat{\theta}$ for large R which, being somewhat analogous to the bootstrap confidence distribution of Hu and Kalbfleish (2000), can be termed as the EF-confidence distribution of θ. Now alternative PE, VE, and IE of θ can be obtained from the EF-confidence distribution as

$$\tilde{\theta} = R^{-1}\sum_r \tilde{\theta}_r\,,\ \tilde{v}(\hat{\theta}) = R^{-1}\sum_r(\tilde{\theta}_r - \tilde{\theta})^2\,, \tag{3.1}$$

and equal-tailed lower and upper end points to obtain $[\tilde{L}_{\alpha/2}(\hat{\theta}), \tilde{U}_{\alpha/2}(\hat{\theta})]$ corresponding to the level $1-\alpha$.

The theory underlying RREE is fairly simple. First observe that for large *n* by the Central Limit Theorem,

$$H_\psi^{-1}(\theta)\psi(y,\theta) = (J_\psi^{-1}(\hat{\theta})H_\psi(\hat{\theta}))^{-1}(\hat{\theta}-\theta) + o_p(1) \sim_{approx} N(0,1) \tag{3.2}$$

Now consider solutions $\hat{\theta}_r$ of the recentered equations $(J_\psi^{-1}H_\psi)^{-1}(\hat{\theta}-\theta)=\varepsilon_r$ where $(J_\psi^{-1}H_\psi)$ is evaluated at $\hat{\theta}$. It easily follows that the empirical distribution $\{\hat{\theta}-\hat{\theta}_r\}_{1\le r\le R}$ given the sample approximates the sampling distribution of $(\hat{\theta}-\theta)$ because $\hat{\theta}_r=\hat{\theta}-(J_\psi^{-1}(\hat{\theta})H_\psi(\hat{\theta}))\varepsilon_r$ by construction. Therefore, the mean, variance, and lower and upper $\alpha/2$ points of the empirical distribution $\{\hat{\theta}-\hat{\theta}_r\}_{1\le r\le R}$ give rise to PE, VE, and IE that are identical (by making R large enough) to those under QL. Now since $\tilde{\theta}_r$ solves $H_\psi^{-1}(\theta)\psi(y,\theta)=\varepsilon_r$, it follows by the asymptotic equivalence in (3.2) that the new RREE empirical distribution $\{\hat{\theta}-\tilde{\theta}_r\}_{1\le r\le R}$ also approximates the sampling distribution of $(\hat{\theta}-\theta)$. Therefore, the RREE empirical distribution $\{\tilde{\theta}_r\}_{1\le r\le R}$ of $\hat{\theta}$ or the EF-confidence distribution of θ gives rise to new PE, VE, and IE that are asymptotically (for large n and R) equivalent to those based on $\{\hat{\theta}_r\}_{1\le r\le R}$, i.e., under QL.

It is important to note that in any application of RREE, it would be better in general to use the nonstudentized pivotal than the studentized one. For example, for $\psi(y,\theta)$ as $\sum_{i=1}^n g_i$ with independent g_i's, it is better in general to use $H_\psi^{-1}\psi(y,\theta)\equiv\left(\sum_i g_i\right)\left(\sqrt{\sum_i g_i^2}\right)^{-1}$ than the studentized version $H_{\psi,t}^{-1}\psi(y,\theta)\equiv\left(\sum_i g_i\right)\left(\sqrt{\sum_i (g_i-\overline{g})^2}\right)^{-1}$ in terms of the validity of the normal approximation; see Godambe and Thompson (1999). Also it may happen that for some ε_r's, the $\tilde{\theta}_r$ values lie outside the admissible range of the parameter θ in which case it is reasonable to use a truncated pivotal distribution to prevent drawing of such ε_r's or use a truncated empirical distribution $\{\tilde{\theta}_r\}_{1\le r\le R}$ after discarding inadmissible $\tilde{\theta}_r$'s. Clearly such truncation would be inconsequential for large n.

It may be noted that the consistency of PE, VE, and IE under RREE for large n and R follows from their asymptotic equivalence to those under the QL method. However, for finite n and R, it might be better to trim the extreme values $\tilde{\theta}_r$ using a suitable rule (such as trimming those lying outside the interval *median +/-2.5(Inter Quartile Range)*) in order to improve the finite sample performance of PE and VE. Any effect of such trimming would be negligible for large n and R. No such trimming is, however, needed for IE because interval boundaries as in quantile estimation are not affected much by extreme values. Finally, we remark that for complex surveys, the EF ψ and its variance V_ψ, unlike the case of simple random sampling or designs ignorable for the model, tend to be quite complicated due to sampling weights in ψ and the need of taking design into account for V_ψ. However, as mentioned earlier, a simple way out might be to use the design effect (τ, say) relative to the simple random sampling or the ignorable design to adjust the variance V_ψ. For example, for the common mean parameter model with variance σ_y^2, if $\psi(y,\theta)$ is $\sum_{i=1}^n w_i(y_i-\theta)$ with sampling weights w_i's, the adjusted variance is obtained as $V_\psi=\tau(\sum_i w_i^2)\sigma_y^2$.

4. **Alternative Method of EF-Bootstrap**

In a pioneering paper on interval estimation, Hu and Kalbfleisch (2000) proposed a novel modification of the classical bootstrap based on EFs as described below. Given independent elementary EFs g_i 's, let $\psi(y,\theta)$ denote $\sum_{i=1}^{n} g_i$, and $\hat{\theta}$ be the solution of $\psi(y,\theta)=0$. Now resample R times with replacement the set $\{g_1, g_2, ..., g_n\}\big|_{\theta=\hat{\theta}}$ to obtain $\{g_{1,r}^*, g_{2,r}^*, ..., g_{n,r}^*\}\big|_{\theta=\hat{\theta}}, 1 \le r \le R$, and define $\psi_r^*(y,\hat{\theta})$ as $\sum_{i=1}^{n} g_{i,r}^*(\hat{\theta})$. Next using recenters ε_r^* based on the bootstrap distribution of the studentized pivotal (defined in the previous section), obtain replicate values θ_r^* as solutions of the following estimating equations for $1 \le r \le R$,

$$H_{\psi,t}^{-1}(\theta)\psi(y,\theta) = \varepsilon_r^* \text{ where } \varepsilon_r^* \equiv H_{\psi_r^*,t}^{-1}(\theta)\psi_r^*(y,\theta)\big|_{\theta=\hat{\theta}} \tag{4.1}$$

and the scaling transformation $H_{\psi,t}(\theta)$ denotes the studentization. Unlike RREE, discarding of recenters leading to inadmissible solutions was, however, not considered, probably because the problem of inadmissibility typically doesn't arise in finding IE. The reason for this is that for EF monotonic in θ which is generally the case, it is sufficient to solve (4.1) for lower and upper cut-offs of the bootstrap distribution $\{\varepsilon_r^*\}_{1\le r\le R}$. However, using the empirical distribution $\{\theta_r^*\}_{1\le r\le R}$, one can also obtain alternate PE and VE as in RREE. Here, as in RREE, it would be useful to introduce truncation and trimming in the interest of improving finite sample performance. Rao and Tausi (2004) also consider use of jackknifing EFs analogous to EF-bootstrap to propose an alternative VE.

We make the following observations in comparing RREE with EF-bootstrap. Unlike RREE, it is better to use the studentized pivotal in EF-bootstrap for IE. The reason for this is that in RREE, the goal is to improve the normal approximation of the EF-based pivotal distribution, while the goal in EF-bootstrap is to improve the approximation of the EF-based pivotal distribution by the empirical or the bootstrap distribution $\{\varepsilon_r^*\}_{1\le r\le R}$. In fact, it was shown by Hu and Kalbfleisch (2000) using Edgeworth expansions that the studentized EF-bootstrap is accurate to second order for IE unlike the nonstudentized one. So for IE, the EF-bootstrap is expected to perform quite well for small samples in comparison to RREE. However, as mentioned in the introduction for VE on the other hand, RREE is expected to perform quite favorably over EF-bootstrap, because VE under EF-bootstrap might remain susceptible to extreme values even after trimming due to discrete nature of the EF-bootstrap distribution. Also as mentioned earlier, unlike EF-bootstrap, application of RREE does not require independent elementary EFs, and thus may be more suitable for survey data. However, both methods are conceptually simple but computationally intensive like any other resampling method.

It might be useful to point out the difference between EF-bootstrap and the classical or C-bootstrap based on EF as defined in Hu and Kalbfleisch (2000). For the C-bootstrap in the EF context, replicate values $\hat{\theta}_C^*$ of $\hat{\theta}$ are obtained by solving

$\psi^*(y,\theta)=0$. In other words, the center of the equation is always set to zero, but the EF on the left hand side changes with the resample. Here, unlike EF-bootstrap, the goal is to approximate the distribution of $\hat{\theta}-\theta$ by the bootstrap distribution $\left\{\hat{\theta}^*_{c,r}-\hat{\theta}, 1\le r\le R\right\}$.

5. **Examples**

5.1 Proportions. It is known (see e.g., Brown, Cai and Dasgupta, 2001) that for estimating proportions (θ) from a random sample, the PE (i.e., the sample proportion) along with its VE under the usual QL method give rise to poor coverage properties of the normality-based IE (also known as the Wald interval) for moderate n even when θ is not too small or large. The main reason for this problem is that variance of the sample proportion depends on θ and the normal approximation with estimated variance doesn't hold very well. A recommended way out is to use the Wilson (1927) interval which is obtained by test inversion based on a suitable EF and its asymptotic normality. This problem was considered by Singh and Dochitoiu (2005) using RREE who showed that the Wilson IE can be obtained from RREE using the EF-confidence distribution. More specifically, here the replicate value $\tilde{\theta}_r$ is obtained by solving

$$\sqrt{n}(\bar{y}-\theta)\Big/\sqrt{\theta(1-\theta)}=\varepsilon_r \tag{5.1}$$

and the solution is given by

$$\tilde{\theta}_r=\left[\bar{y}\left(\frac{n}{n+\varepsilon_r^2}\right)+\frac{1}{2}\left(\frac{\varepsilon_r^2}{n+\varepsilon_r^2}\right)\right]-\varepsilon_r\sqrt{\frac{1}{n+\varepsilon_r^2}\left[\bar{y}(1-\bar{y})\left(\frac{n}{n+\varepsilon_r^2}\right)+\frac{1}{4}\left(\frac{\varepsilon_r^2}{n+\varepsilon_r^2}\right)\right]} \tag{5.2}$$

Note that the Wilson interval is simply given by the set $\left\{\theta:\left|\sqrt{n}(\bar{y}-\theta)\Big/\sqrt{\theta(1-\theta)}\right|\le z_{\alpha/2}\right\}$ where $z_{\alpha/2}$ denotes the upper cut-off point of the normal distribution. It should be noted that truncation of the RREE pivotal distribution is not needed because of the lack of the problem of inadmissible solutions. The RREE method also provides alternative PE and VE with favorable properties in comparison to the traditional estimators. For complex surveys, the above approach goes through except that the variance $\theta(1-\theta)/n$ is multiplied by the design effect; see also Kott, Andersson, and Nerman (2005) for a related reference on IE from survey data.

5.2 Ratio of Means. Consider the estimation problem of ratio of means ($\theta=\mu_y/\mu_x$) of variables y and x from a random sample of observations $(y_i,x_i)_{1\le i\le n}$. Under the usual QL approach based on the EF defined by the difference between the sample mean ($\bar{y}$) of y and the unknown ratio (θ) times the sample mean ($\bar{x}$) of x, the IE based on the Taylor linearized VE (also known as the Wald method) is known to perform poorly. It is in the sense of undercoverage of IE for moderate or small samples probably due to skewness in the distribution of ratio of sample means. An improved IE was proposed by Fieller (1932) using a normal EF-based pivotal. However, the Fieller IE is known to be not sufficiently conservative. Besides IE, it would also be useful to have an alternative to the linearized variance estimator which is generally more stable for small samples. This problem was investigated by Singh and Tam (2007) using RREE who showed that the Fieller IE corresponds to the use of studentized EF-based pivotal in RREE without any truncation for inadmissible values. However, it could be improved by using nonstudentized pivotal along with truncation of the pivotal distribution. More specifically, denoting the PE $\bar{y}/\bar{x}$ by $\hat{\theta}$, the Taylor linearized VE is given by

$$v_{TL}(\hat{\theta}) = \hat{\theta}^2(c_{\bar{y}\bar{y}} + c_{\bar{x}\bar{x}} - 2c_{\bar{y}\bar{x}}) \text{ where } c_{\overline{yx}} = (\overline{yx})^{-1}(s_{xy}/n) \tag{5.3}$$

and $s_{yx} = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})$. The terms $c_{\overline{yy}}$ and $c_{\overline{xx}}$ are similarly defined and denote squared coefficients of variation. The Wald's IE is based on the approximate normal pivotal $(\hat{\theta}-\theta)\big/\sqrt{v_{TL}(\hat{\theta})}$ which may have serious undercoverage for small or moderate n along with highly unbalanced tail coverage. Fieller (1937) proposed to use the studentized normal pivotal and obtained interval boundaries as nonlinear functions by solving

$$(\bar{y}-\theta\bar{x})\Big/\sqrt{\sum[(y_i-\theta x_i)-(\bar{y}-\theta\bar{x})]^2/n(n-1)} = \pm z_{\alpha/2} \tag{5.4}$$

Note that for RREE, the nonstudentized pivotal $(\bar{y}-\theta\bar{x})\Big/\sqrt{\sum(y_i-\theta x_i)^2/n^2}$ is used to obtain the empirical distribution $\left(\tilde{\theta}_r\right)_{1\le r\le R}$ after discarding inadmissible $\tilde{\theta}_r$'s corresponding to ε_r's that do not satisfy the admissibility condition for each given sample. The condition is

$$1-\varepsilon_r^2(n^{-1}+n^{-1}(n-1)c_{\overline{xx}}) > 0 \tag{5.5}$$

Now IE is obtained as before from the empirical distribution $\left(\tilde{\theta}_r\right)_{1\le r\le R}$. It is found to have improved coverage over Fieller's. PE and VE are also obtained after suitable trimming of extreme values.

5.3 Quantiles. The QL method for IE of the qth quantile (θ_q) relies on linearization via Bahadur representation of the EF-pivotal $H_{\psi}^{-1}\psi(\mathbf{y},\theta_{\mathbf{q}}) \equiv (\hat{F}(\theta_q)-q)\big/\sqrt{q(1-q)/n}$ of (3.2) where $\hat{F}(.)$ is the empirical distribution function based on a random sample of size n. However, the analogue of the observed information J_{ψ} of (2.1) involves the unknown density function and the resulting normality-based IE is known to have poor coverage properties for small samples. On the other hand, the popular Woodruff's (1952) method for IE works surprisingly well (Sitter and Wu, 2001) although developed in a somewhat mysterious way by treating θ_q known and $F(\theta_q)$ unknown (reverse of what is actually given), and then use the above EF with q in the denominator replaced by $\hat{F}(\theta_q)$ to obtain the usual normality-based IE. Next θ_q is treated as unknown but in the denominator θ_q is estimated by $\hat{\theta}_q$ to have a known variance term, and the above interval is then inverted to obtain an IE for θ_q. It is interesting to note that the Woodruff's method essentially uses the same EF-pivotal as the proposed method of RREE because $\hat{F}(\hat{\theta}_q) \doteq q$. In RREE for quantiles, inadmissible values of $\tilde{\theta}_{q,r}$ are, however, discarded before using the empirical distribution $(\tilde{\theta}_{q,r})_{1\le r\le R}$ for IE. For complex designs, the variance expression in the denominator of the above EF-pivotal can be multipled by a suitable known design effect such as an average estimated design effect over several neighboring values of q as considered by Singh and Phillips (2007). One could also use a studentized EF-pivot as was proposed by Francisco and Fuller (1991) which basically entails replacing known q in the denominator by unknown $\hat{F}(\theta_q)$. Thus a studentized version of RREE is essentially equivalent to Francisco-Fuller's method except for the truncation of the EF-confidence distribution in RREE. In terms of VE, the RREE method can be used to obtain as before an alternative estimate to the Bahadur-linearized VE which is subject to instability due to the required estimation of the density function at $\hat{\theta}_q$. Another alternative, which is quite novel and widely applicable, was proposed by Francisco and Fuller (1991) and consists of using the squared half adjusted width of the IE. In other words, the IE is treated as a normality-based interval and the adjustment factor to the half-width is simply inverse of $z_{\alpha/2}$ for a given level $1-\alpha$. Clearly the adjustment factor depends on the level chosen in a somewhat ad hoc manner but the value of 95% seems to have worked well in simulation studies of Rao and Wu (1987).

5.4 Logit Model for Mean Parameters. For logit models, the QL method is commonly used for estimating model parameters which are, in turn, used to estimate the outcome prevalence for a given set of covariates. In practice, often the logit-Wald method is used to obtain asymmetric IE which is deemed preferable due to skewness in the distribution of estimates of low or high prevalence outcomes. Moreover, the logit-Wald method ensures that the iE boundaries are in the desired range. For this method, the IE is first constructed in the logit scale using the approximate normality of model parameter estimates under QL (this is the Wald step), and then the boundaries of IE are transformed back to the original scale via the inverse logit function. Similar to other problems discussed earlier, the Wald IE is expected to perform poorly for small or moderate samples due to nonlinearity of model parameter estimates, and the RREE method might be able to improve the logit-Wald based IE. This is an interesting problem as it involves the application of RREE to a multi-parameter case and is currently being investigated; see Singh and Nadeau (2008). For example, for a two parameter logit model for the mean μ_i of the ith observation with a single covariate x_i, the mean in the original scale is given by

$$\mu_i(\theta) = \exp(\theta_0 + \theta_1 x_i)[1 + \exp(\theta_0 + \theta_1 x_i)]^{-1} \tag{5.6}$$

and the corresponding EFs based on a random sample of n observations are

$$\psi_1(y,\theta) = \sum_{i=1}^{n} (y_i - \mu_i(\theta)), \quad \psi_2(y,\theta) = \sum_{i=1}^{n} x_i(y_i - \mu_i(\theta)) \tag{5.7}$$

Denoting by $H_\psi(\theta)$, as before, the Cholesky root of the covariance matrix $V_\psi(\theta)$ of the EF-vector ψ, the replicate values of the vector parameter estimate $\hat{\theta}$ are obtained by solving iteratively

$$\begin{pmatrix} \psi_1(\theta) \\ \psi_2(\theta) \end{pmatrix} = H_\psi(\theta) \begin{pmatrix} \varepsilon_{1,r} \\ \varepsilon_{2,r} \end{pmatrix} \tag{5.8}$$

where $\{\varepsilon_{1,r}, \varepsilon_{2,r} : 1 \le r \le R\}$ are independent standard normal deviates. Using (5.6), the EF-confidence distribution of the prevalence parameter $\mu(x,\theta)$ for a given covariate level can be obtained from that of the parameter vector θ. Now, alternative PE,VE, and IE can be constructed as before.

6. Simulation Results

We report results from a small simulation study to illustrate comparative results for RREE and EF-bootstrap as alternatives to the commonly used QL method. For this purpose, we consider the problem of ratio of means for two cases: symmetric population distribution and skewed population distribution. For the first case, simple random samples of size n(=20) were drawn from a bivariate normal distribution with means $\mu_y = 5, \mu_x = 2$, and covariance matrix defined by $\sigma_y^2 = \sigma_x^2 = 1, \sigma_{yx} = 0.2$. Thus the parameter of interest $\theta = 5/2$. For each simulation m= 1, 2,…10,000, R(=5000) recenters were generated for both RREE and EF-bootstrap methods. For the second case, samples were generated from a bivariate log-normal distribution such that in the log scale the variables y and x have the same mean and covariance as in the first case. The results are summarized in Tables 1 and 2.

In terms of bias and MSE of PE and ME (denotes the MSE estimator which is identical to VE) for the case of symmetric population, both QL and RREE behave very similarly except for the relative bias (RB) of PE and ME. It is interesting to note that here the linearization VE (or ME) under QL is least biased and fairly stable in comparison to others even for the sample size of 20. Both PE and ME for RREE are somewhat biased upward compared to QL but the relative root MSE (RRMSE) are at par with that for QL. The EF-Boot, on the other hand, behaves at par with RREE for PE, but for ME, it is considerably biased upward and the corresponding RRMSE is also somewhat higher. For the skewed population case, however, again QL and RREE behave similarly for the most part except that RB in PE for RREE is almost 36% higher than that for QL, but RB in RRMSE for RREE is about 30% lower. The results for EF-Boot suggest that both PE and ME are highly biased as well as the corresponding RRMSE are considerably higher indicating instability in both PE and ME for EF-Boot.

In terms of comparison of coverage probabilities and average length, it is observed from Table 2 that for the symmetric case, both RREE and EF-Boot methods behave well and have similar performance, but QL shows a slight undercoverage in the central part with highly under- and over- coverage respectively for left and right tails. The average length of IE for QL tends to be shorter but this is not of much comfort due to bias in coverage probabilities. Turning now to the skewed case, the QL method as expected does quite poorly in terms of coverage probabilities. However, both RREE and EF-Boot perform reasonably well and almost at par except that the EF-Boot has somewhat better coverage in the tails. Note that the direction of bias for RREE and EF-boot are in opposite directions in central and tail parts of IE. The average length for EF-boot is similar to that for RREE but the standard deviation for EF-boot is seen to be considerably higher.

7. Summary and Concluding Remarks

In this paper, a new method termed randomly recentered estimating equations (RREE) was introduced to deal with the problems of instability of variance estimation and coverage bias of the interval estimation under the quasi-likelihood (QL) method for small or moderate samples. It was observed that RREE is simple to implement like the EF-bootstrap method of

Hu and Kalbfleisch (2000) but is applicable more generally to the case of dependent observations. This implies that the RREE method may be more suitable for survey data. The method RREE is based on an improved normal approximation to the EF-based pivotal unlike the improved bootstrap approximation to the EF-based pivotal used for EF-Bootstrap. It may be remarked that although the normal approximation to the EF-pivotal used by RREE is considerably improved as suggested by the simulation study, it is accurate to only first order. It is possible, following Wang and Taneichi (2006), to make it second order accurate by using a suitable normalizing transformation to EF that depends on third moments of elementary EFs to correct for skewness. Use of such transformed EF should further improve the performance of RREE and should be investigated in future. However, for survey data, this may not be very practical, due to possible difficulty in obtaining reasonable estimates of third moments of the EF.

The RREE methodology goes through for the multi-parameter case although it gets computationally more tedious as in the case of EF-Bootstrap. In this context, use of Gauss-Seidel type algorithms can be used to simplify computational complexity. Also as with EF-Bootstrap, nuisance parameters can be handled by working with an EF-pivotal which is either free from nuisance parameters or whose distribution doesn't depend (approximately for large n) on nuisance parameters. Finally, we note that as suggested by Léger (2000) in his discussion of EF-Bootstrap, it would be useful to study a robustified version of RREE based on EF corresponding to M-estimation. This problem is currently under investigation.

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Table 1 Bias and MSE of Point and MSE Estimators

	n = 20			
	PE		ME	
Method	**RB (%)**	**RRMSE (%)**	**RB (%)**	**RRMSE (%)**
Symmetric				
QL	1.2	11.8	1.2	62.0
RREE	1.9	12.1	3.1	61.8
EFBoot	2.0	12.2	8.7	66.9
Skewed				
QL	6.6	40.8	-12.4	161.2
RREE	9.0	41.6	-8.7	173.9
EFBoot	12.5	50.0	57.2	1065.2

Table 2 Coverage Probabilities of Interval Estimates

	n = 20				
Method	L(2.5%)	C(95%)	R(2.5%)	Avg(Len)	SD(Len)
Symmetric					
QL	0.9	93.8	5.3	1.118	0.314
RREE	2.3	95.5	2.2	1.291	0.399
EFBoot	2.7	95.0	2.3	1.289	0.408
Skewed					
QL	1.3	89.3	9.4	3.588	1.927
RREE	1.6	96.6	1.8	4.951	2.750
EFBoot	3.0	94.1	2.9	5.260	5.596

M = 10,000; R = 5,000