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Mathematical Derivation of the Total Requirements Tables for Input-Output Analysis[1]

From make and use tables, the following are defined:

^: A symbol that, when placed over a vector, indicates a square matrix in which the elements of the vector appear on the main diagonal and zeros elsewhere.

q: A column vector in which each entry shows the total amount of each commodity's output.

g: A column vector in which each entry shows the total amount of each industry's output.

U: Intermediate portion of the use matrix in which the column shows for a given industry the amount of each commodity it uses, including noncomparable imports and used and secondhand goods. This is a commodity-by-industry matrix.

V: Make matrix in which the column shows for a given commodity the amount produced in each industry. This is an industry-by-commodity matrix. V has columns showing only zero entries for noncomparable imports and used and secondhand goods.

B: Direct input coefficients matrix in which entries in each column show the amount of a commodity used by an industry per dollar of output of that industry. This is a commodity-by-industry matrix.

$$B = U\hat{g}^{-1} \qquad (1)$$

D: A matrix in which entries in each column show, for a given commodity, the proportion of the total output of that commodity produced in each industry. In the model, it is assumed that each commodity is produced by the various industries in fixed proportions. This is an industry-by-commodity matrix. D is also referred to as the market share matrix or transformation matrix.

$$D = V\hat{q}^{-1} \qquad (2)$$

i: Unit (summation) vector containing only l's.

I: Identity matrix, where I = $\hat{i}$.

e: A column vector in which each entry shows the total final demand purchases for each commodity from the use table.

From the above definitions, the following identities are derived:

$$q = Ui + e \quad (3)$$

$$g = Vi \quad (4)$$

The model expressed in equations (1) through (4) thus involves two constants (*B, D*) and five variables (*U, V, e, q, g*). The model solution is derived as follows:

From (1) and (3), we derive:

$$q = Bg + e \quad (5)$$

From (2) and (4), we derive:

$$g = Dq \quad (6)$$

Substituting (6) into (5) and solving for *q* gives:

$$q = B(Dq) + e$$
$$(I - BD)q = e$$

$$q = (I - BD)^{-1} e \quad (7)$$

The matrix $(I - BD)^{-1}$ is known as the commodity-by-commodity total requirements matrix and it shows, on a per-dollar basis, the commodity output the economy generates in order to provide commodities to final users.

Substituting (5) into (6) and solving for *g* gives:

$$g = D(Bg + e)$$

$$(I - DB)g = D e$$

$$g = (I - DB)^{-1} De \quad (8)$$

The matrix $(I - DB)^{-1}$ is known as the industry-by-industry total requirements matrix and it shows, on a per-dollar basis, the industry output the economy generates in order to provide an

industry’s commodities to final users. The vector De is a final demand vector where each entry shows the final demand for an industry’s output.

Substituting (7) into (6) and solving for q gives:

$$g = D(I - BD)^{-1}e \qquad (9)$$

The matrix $D(I - BD)^{-1}$ is known as the industry-by-commodity total requirements matrix and it shows, on a per-dollar basis, the industry output the economy generates in order to provide commodities to final users.

BEA’s published total requirements tables are calculated using a hybrid approach between the industry-technology assumption (ITA) and the commodity-technology assumption (CTA) methods. The ITA proposes that all commodities made by an industry share the same input structure. In contrast, the CTA proposes that each commodity has a unique input structure that is independent of the producing industry.[2]

Specifically, BEA’s method involves a two-step process. The CTA is first applied to secondary products produced by an industry. Secondary products are redefined when the input structure for a secondary product of an industry differs significantly from the input structure for the primary product of that industry. The output of the secondary product is moved to an industry in which production of that product is primary. After a product is redefined, the associated inputs are then moved, or reallocated. These redefinitions and reallocations form the basis for the after redefinition make and use tables and provide a more homogeneous relationship between input structure and products.[3]

From these supplementary tables, as the second step of the hybrid approach, the ITA is followed in the mathematical derivation of total requirements tables. Under the ITA, the best estimate of the inputs used for secondary products comes from the producing industry. The calculation of the direct requirements table, the foundation of the total requirements tables, implicitly follows this assumption by estimating the inputs related to the secondary products as a portion of the total inputs to the producing industry.

As a result of utilizing this hybrid approach, the total requirements tables are a more useful and effective tool for analyzing the interrelationships among industries and the relationships between industries and the commodities they use and produce.

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[1] The notation and derivation of the tables presented follow the *System of National Accounts* recommended by the United Nations. See: *A System of National Accounts Studies in Methods*, Series F No. 2 Rev. 3, United Nations, New York, 1968; also, Stone, R., Bacharach, M. & Bates, J., "Input-Output Relationships, 1951-1966," *Programme for Growth*, Volume 3, London, Chapman and Hall, 1963.

[2] US Department of Commerce, Bureau of Economic Analysis, Guo, J., Lawson, A. and Planting, M. “From Make-Use to Symmetric I-O Tables: An Assessment of Alternative Technology Assumptions” http://www.bea.gov/papers/pdf/alttechassump.pdf

[3] U.S. Department of Commerce, Bureau of Economic Analysis, Horowitz, Karen J. and Planting, Mark A. “Concepts and Methods of the U.S. Input-Output Accounts” http://www.bea.gov/papers/pdf/IOmanual_092906.pdf